Appendix A

Definitions, Proofs, and Mathematical Language

Mathematics does not grow through a monotonous increase of the number of indubitably established theorems but through the incessant improvement of guesses by speculation and criticism, by the logic of proofs and refutations.

–Imre Lakatos

You know that I write slowly. This is chiefly because I am never satisfied until I have said as much as possible in a few words, and writing briefly takes far more time than writing at length.

–C. F. Gauss

A good mathematician can achieve understanding. A great mathematician can pass it on.

–Ben Orlin (*Math with Bad Drawings*)

A.1. Writing mathematics

Writing mathematics is not a riddle to find the correct sequence of words; it is a skill of communicating your mathematical ideas. Learning to write mathematics is much like learning to play bridge—there are strategies (that you might discover for yourself with enough experience and reflection), and there are conventions (that are very helpful in practice, but somewhat arbitrarily decided upon). Depending on who you talk to, there are sometimes various—and perhaps even contradictory!—conventions.[1] These are all in the service of making your writing *clear*—i.e., getting your ideas and arguments to transfer to the reader's mind as faithfully as possible. This is true whether you are writing a solution, defining a mathematical idea,

[1]But "conventions are important, and lives can depend on them. We know this whenever we take to the highway. If everyone else is driving on the right side of the road (as in the U.S.), you would be wise to follow suit." –Steven Strogatz

proving a theorem, or explaining an intuition, though these different contexts of mathematical writing also have different conventions associated to them.

Here is a (certainly incomplete) set of things to keep in mind when writing mathematics in any context[2]:

(1) **Mathematical writing is writing.** When writing mathematics you must follow all of the usual rules of spelling, grammar, punctuation, sentence structure, etc. Even equations that are set off, such as

$$\int_0^4 x^2\,dx = \frac{4^3}{3},$$

are part of a sentence and should be punctuated appropriately. You should be able to read out loud anything you write (including notation) in complete sentences. For example,

$$\forall\ \epsilon > 0\ \exists\ K > 0 \text{ s.t. } k > K \Longrightarrow |a_k - L| < \epsilon$$

is actually a complete sentence that reads: "for all epsilon greater than zero there exists big-kay greater than zero such that small-kay greater than big-kay implies that the absolute value of a-sub-kay minus ell is less than epsilon."

(2) **Know your audience.** How you write—how many details you include, what notation you use, what knowledge your explanations assume—depends on who your audience is. In a mathematics course, for instance, your audience is your classmates (not your professor or a high school student).

(3) **Use an appropriate amount of detail.** The term "appropriate" is relative to your audience. You can spare your audience a step-by-step demonstration of something they already know well, though it can be a good idea to refer to the procedure. For example,

> *Too many details:* In order to solve the equation $x^2 - 4x - 12 = 0$, we factor the left hand side in the form $(x + A)(x + B)$ using the fact that the product of A and B must be -12 and the sum of A and B must be -4. This happens for $A = -6$ and $B = 2$, so we get $(x - 6)(x + 2) = 0$. Finally, since the product of two numbers is 0 if and only if one of the two numbers is zero, we have $x - 6 = 0$ or $x + 2 = 0$, which is the same as $x = 6$ or $x = -2$.
>
> *A better amount of details:* In order to solve the equation $x^2 - 4x - 12 = 0$, we factor the polynomial and set each term to 0. This yields $x = 6$ or $x = -2$.

Of course, in a different context the latter could be considered too brief and the former a more appropriate amount of details.

(4) **Be concise and understandable.** Part of the utility of mathematics is its ability to separate ideas of structural importance from those that are particular to certain situations. One goal in writing mathematics is to indicate this distinction by taking away superfluous ideas and explanations. This is particularly important in writing proofs, where short, clear arguments are highly

[2]This list was inspired by "The nuts and bolts of writing mathematics" by David Richeson, another excellent resource on mathematical writing that you may find helpful.

valued. Another goal of writing mathematics, though, is to be understandable. For example, outside of proofs, much of our "meaning-making" comes from illustrative examples, intuitions, analogies, applications, etc. Using these well is a form of art, and subject to many revisions.

(5) **Reread and edit what you have written.** Forming and communicating ideas is partly like making a clay sculpture (an additive process) and partly like sculpting marble (a subtractive process). When you form ideas you should add them to your work in writing, but they very likely will need to be chiseled, reformed, and rearranged. Look back at what you wrote a day later or a week later or a month later: Does your writing say what you wanted it to? Are there ideas that you thought were important but in retrospect are not necessary? Or ideas that you said little about but now realize are a big deal? Did you encounter new ideas that help you refine your previous work? Revision is an important part of creating mathematics.

A.2. Writing definitions

Definitions are cornerstones of mathematics—all subsequent propositions and theorems about a concept come in reference to the definition of that concept, and they determine the shape of the entire structure. If you want a beautiful building, you need to have a good quality cornerstone. Crafting a definition, however, is a also dynamic process in which you might make multiple revisions of a definition as you gain experience about how you want to use it and how you want it to relate to other ideas. Here are three criteria to test a definition against:

(1) A good definition should *apply the way you want it to.* Check how the proposed definition applies to examples you have in mind. Does it do what you want it to do?[3]

(2) A good definition should be *clear*. Suppose you give the proposed definition to a classmate and ask them to apply it to examples. Can they tell how to use the definition? Do they get the results you get when applied to the sequences in the previous investigation?[4]

(3) A good definition should be *concise*. Is there any way you can condense the proposed definition? Is all of the notation necessary, or can you use less?[5]

A.3. Writing proofs

A mathematical proof carefully builds new knowledge based on previous knowledge and agreed-upon rules. This means that "every mathematical statement in a proof must be justified in one or more of the following six ways: by an axiom; by a previously proved theorem; by a definition; by hypothesis (including [as a special case] an assumption for the sake of contradiction); by a previous step in the current

[3] Note that a good definition can also change your understanding of an example you had in mind...

[4] Of course, good definitions may also take time to understand well, so you should see if differences in application of the definition can be resolved by the written definition.

[5] Sometimes a good definition can be made shorter in a way that slightly changes the idea, and you have to decide which one best "matches" what you think the term should mean.

proof; or by the rules of logic”[6]. In this section, we briefly consider rules of logic, quantifiers and negation, and the particular technique of proof by contradiction.

A.3.1. Logic.

Definition A.1. Consider the logical statement “if P then Q” (this can be written in logic notation as P $\Rightarrow$ Q). A couple of related logical statements are:

STATEMENT	“If P then Q.”	$P \Rightarrow Q$
CONVERSE	“If Q then P.”	$Q \Rightarrow P$
CONTRAPOSITIVE	“If not Q, then not P.”	$\neg Q \Rightarrow \neg P$
INVERSE	“If not P, then not Q.”	$\neg P \Rightarrow \neg Q$

We consider the statement “if P then Q” true if Q is true whenever P is true (cases in which P is *not true* has no bearing on the truth of this statement). The statement “if P then Q” is false if there is a case in which P is true but Q is not true. This means that the statement “if Jake bowls a 305, then he will be elected president” is a logically true statement, since bowling a 305 is impossible and so “Jake bowls a 305” is never true. Sometimes such statements are described as “vacuously true.”

***Investigation A.2.** This problem asks you to investigate the truth values of some of the related logical statements listed above.

(1) Write an “if P then Q” statement that is true but whose converse is false. (Your statement does not have to be mathematical.)

(2) Write the contrapositive of your statement. Is the contrapositive true or false?

(3) Is there a statement that is true (resp. false) but whose contrapositive is false (resp. true)? If so, write one down. If not, explain why.

(4) Earlier in this text, you proved that if p is a prime number, then $\sqrt{p}$ is not rational. What is the contrapositive of this statement?

A.3.2. Quantifiers and negation. Many of our definitions and results in this class involve quantifiers (we have already encountered these in a number of contexts).

Definition A.3. “For all” is called the universal quantifier, denoted symbolically by $\forall$. “There exists. . . such that” is called the existential quantifier, denoted symbolically by $\exists$.

It often happens that multiple quantifiers show up in the same statement.

***Exercise A.4.** There are often nested quantifiers in mathematical definitions and theorems. In some cases, the order of nested quantifiers makes a difference. How are the two following statements different?

(1) $\exists\ M > 0$ such that $\forall\ k$, $a_k \leq M$.

(2) $\forall\ k$, $\exists\ M > 0$ such that $a_k \leq M$.

[6] John M. Lee, “Some Remarks on Writing Mathematical Proofs.”

Order also makes a big difference when negating a statement. Consider the following tagline from the 1960's TV show *You Don't Say!*: "It's not what you say that counts, it's what you DON'T say." The placement of the word "not" makes a big difference! We use negations in mathematics quite a bit (e.g., a sequence that diverges is one that *does not converge*, a sequence that is unbounded is one that is *not bounded*). Here is a formal definition:

Definition A.5. The negation of a statement P is defined to be a statement ¬P such that P is true ⇔ ¬P is false.

If a statement is notationally more complicated than "P", we will put parenthesis around the negated statement—e.g., ¬(P and Q).

There are often many possible forms of a negation. For example, the statement "all pigs do not fly" can be negated as "not all pigs do not fly," but also "there is a pig that does fly.'

***Problem A.6.** Find a few equivalent forms of the negation of the provided statement. Can you find a form of the negation that does not involve the word "not"?

(1) $\forall\, x,\, x > 3$.
(2) $\forall\, p$, either p is prime or p is even.[7]
(3) There exists a car all of whose parts are made in Michigan.
(4) All pianos have more white keys than black keys.

While a negation may have multiple equivalent forms, you may find that the form eschewing the word "not" is particularly nice. This is especially the case if you are interested in demonstrating the negation—this preferred form gives a positive description of what you should look for.

The process of finding this "preferred" form of the negation of a statement with quantifiers has a certain algorithmic feel to it. See if you can discover the rules in the following example.

***Problem A.7.** Consider the statement

"In every state there is a county in which every town has fewer than 10,000 residents."

(1) Write down the statement using the quantifiers "∀" and "∃... such that".
(2) The following are five equivalent forms of the negation. Write each one down using quantifiers (as above) and the negation symbol ¬(...), with the entire phrase that is negated inside the parenthesis.
    (a) "It is not the case that in every state there is a county in which every town has fewer than 10,000 residents."
    (b) "There is a state that does not have a county in which every town has fewer than 10,000 residents."
    (c) "There is a state in which every county does not have only towns with fewer than 10,000 residents."
    (d) "There is a state in which every county has a town with not fewer than 10,000 residents."

[7] Note that "or" in mathematics is inclusive: "A or B" includes the case "A and B".

(e) "There is a state in which every county has a town with at least 10,000 residents."

Complete the following statements with the appropriate quantifier (in the process, convince yourself that the theorem is true; we will not take the time to rigorously prove this):

***Theorem A.8.** Let $P(x)$ be a statement about x. The following are logical equivalences:

(1) $\neg$ ($\forall$ x, $P(x)$) is equivalent to _______, $\neg P(x)$.
(2) $\neg$ ($\exists$ x such that $P(x)$) is equivalent to _______, $\neg P(x)$.

***Problem A.9.** Here's a challenge: negate the following statement (with three nested quantifiers!) about the sequence (a_n):

"For all $\epsilon > 0$, there exists an N such that for all $n > N$, $|a_n - L| < \epsilon$."

What does the negation mean?

A.3.3. Proof by contradiction. Some styles of proof come with certain structures and conventions. One of the most common (and, indeed, most useful) is *proof by contradiction*.

What makes this method so useful? Suppose you want to prove something like "every boat has a hull." But you are not sure what to do because you can't actually see most hulls of boats and tackling "all boats" seems overwhelming. Instead of trying to describe an arbitrary boat (impossible!) and show it has a hull, you can suppose (for the sake of contradiction) that there is a boat without a hull and show that this cannot possibly work. Indeed, without a hull, the boat would take on water and sink, a clearly untenable situation for a boat. Your conclusion is thus that every boat must have a hull.

Proof by contradiction allows you to use all of the assumptions of the theorem statement *as well as the negation of the conclusion*. (You have a boat *and* it does not have a hull.) In return, all you need to show it that this is an impossible situation—i.e., this mix of assumptions cannot work. The conclusion of this is that, in the presence of the theorem's assumptions, the negation of the conclusion must be false—which means the conclusion must be true!

Two pieces of advice about writing a proof by contradiction:

(1) Reading a proof by contradiction can be jarring if it is not clear it is a proof by contradiction. The writer and lecturer Dale Carnegie gave the following advice on giving a speech: "Tell the audience what you're going to say, say it; then tell them what you've said." This is also good advice for clearly framing a proof by contradiction.
(2) Because it can be so useful to have an additional assumption (the negation of the conclusion), you may craft a proof by contradiction in cases where it is not necessary. After you have written a proof by contradiction, always read it over again to see if the same argument could be used in a "direct proof" format (i.e., without assuming the negation of the conclusion). If so, rewrite your proof in the direct form.