Preface

Explorations in Topology and Analysis is both an introduction to the basic notions of real analysis and related topics as well as to the practice of mathematics as is done in upper-level courses. It is intended for students who have seen some calculus before (perhaps in high school) and are motivated to learn the subject at a deeper level, and who are generally interested in having a rich intellectual experience in mathematics. The text is organized around students' engagement with the material: investigations to explore, theorem statements to prove, and exercises and problems to solve. Definitions, Theorems, Investigations, and Exercises that are meant to be completed by the reader are marked by an asterisk $*$.

This text grew out of our teaching Math 176, an inquiry based learning course for first-year students, sponsored by the Center for Inquiry Based Learning at the University of Michigan. As in all other IBL courses taught at Michigan, worksheets were the main tool we used in the classroom. These consist of a selection of the problems contained in the text, and are to be worked on by the students in groups of three of four. Students take turns presenting their work at the end of class, and this is followed by a class discussion. The role of the instructor in the classroom is to introduce the material the students will work on a given day, to facilitate group work, and to guide class discussion during the presentations. Outside of the classroom, the instructor's work included preparing the worksheets, guided by the progress (or lack thereof) or the direction that the discussion took in the previous class. We regularly assigned homework, and there were two mid-term and a final examinations. Many 176 students go on to take Honors Multivariable Calculus.

The main role of this book, as we currently envison it, is twofold. First, it provides the material for worksheets for classroom use, and second, it tells a mathematical narrative. We hope that this book will help the students see a bigger picture than what they are able to see when they work one worksheet at a time. Of course the instructor shares the responsibility for this as well.

The mathematical content of the book is the underpinnings of Calculus of a single variable. We start with the standard set of axioms for an ordered field, and the Axiom of Completeness in the form that any sequence of nested closed intervals in $\mathbb{R}$ has a non-empty intersection. Throughout the book we steer students to use three mathematical tools in constructing their proofs: bisection, iteration, and nested intervals. The first time these tools are used in tandem is for the Monotone

Convergence Theorem, whose proof is given in the text as a kind of blueprint for how these tools can be leveraged together. For many of the other big theorems, such as the Bolzano-Weierstrass Theorem and the Intermediate Value Theorem, the text provides some guidance but asks the student to complete the proof. Note that these proofs are not intended to be done individually by students, but rather in groups and with the entire class as resources in making headway.

We have inlcuded much more material than what we have been able to cover in one 14-week term. A traditional first course in real analysis would cover much of the content of Chapters 1, 2, 3, and 4, but in more depth and in a more mathematically formal way. This course moves slower than a real analysis course would, in part because we are asking students to develop formality gradually and via their intuition.

Chapter 1 is an introduction to real numbers and sequences. Chapter 2 applies these tools to open, closed, compact sets and continuous functions. It also contains a section on extending these ideas to the plane $\mathbb{R}^2$. Chapter 3 covers highlights (from a pure mathematician's point of view) of differential calculus; there are some computational problems, but the emphasis is still on definitions, theorems, and proofs. Chapter 4 covers highlights of integral calculus.

Chapter 5, on dynamical systems, can be used either as diversions throughout the text or on its own. It follows in the same style as the rest of the text, both in how the student interacts with it and also with respect to the (bi)section, iteration, and nested intervals theme. Much of the material in this chapter has been used for a summer enrichment course in the Michigan Math and Science Scholars program.

On rigor. One of the aims of the book is to introduce the students to rigorous mathematical language and argumentation. This is of course not an easy task! Our approach has been to proceed gradually, and to use the opportunities provided by student discussion in class to introduce the basic logical elements as they are needed. At first, a mixture of quantifiers, other logical symbols, and English language is used. For example, in the first definition of convergence of a sequence (a_k), we are happy to arrive at: $\lim a_k = L$ if

$$\text{for all } \epsilon > 0, \qquad |a_k - L| < \epsilon \text{ for all sufficiently large } k.$$

As the students work with this definition, the need for specifying the precise meaning of "all sufficiently large k" arises naturally. Only then is the second quantifier "$\exists\ K$ such that..." introduced. It has been our experience that, in the course of the semester, under the instructor's guidance the students themselves will increasingly come to demand that precise language be used in class.

To the Student. This text is an invitation to practice mathematics. Not the kind of practice in which you repeat a procedure that you read about or were shown; rather, you will be practicing the discipline of mathematics. A mathematician asks questions, investigates new areas, tries out ideas (that often don't work), writes down progress, revises their work, and builds on their expanding knowledge. This book is structured to guide you in that process.

Explorations in Topology and Analysis covers the foundational topics of calculus from a mathematician's point of view. While there are many fascinating and useful applications of calculus, this text is a study of the definitions, tools, and their relationships that create the backbone of calculus. In pursuit of that, it empasizes:

1. The central importance of *ideas*. We will consider technical competence a byproduct of a thorough investigation of ideas, rather than a basic skill to start forming ideas with.

2. The central importance of *questions*. An important role of a mathematician is to ask good questions. This text will ask many questions for you to work on—these are meant to form a basis for your own inquiries. That is, you should make these questions your own by working on them and refining and deepening your ideas about them—and by asking further questions (of yourself, of your classmates). This is a style of *inquiry based learning*—learning that starts with your inquiries (either self-formed or by making a question your own) and is set into place by your own investigations.

3. The central importance of *the process of investigation*. In particular, we wish to investigate ideas in the style of mathematicians in order to create and refine deductive, quantitative arguments. While we hope you will be proud of the end product, the main goal is to develop some of the logical and quantitative competencies involved in taking ideas/intuition/questions and refining them into deductive-reasoning validated products.

This book is designed to be implemented with a guide—someone with mathematical experience and sophistication who will be able to help you navigate the process of discovery and exploration. The open-ended nature of mathematical inquiry can be both exhilarating and debilitating, and forming and recognizing "good" questions is an acquired skill (—one you will be working on acquiring!). A guide will help you strike a balance between focus and free exploration, recognize when to work through roadblocks and when to abandon an unproductive line of inquiry, and how to learn from your work.