

A FRANKS' LEMMA FOR CONVEX PLANAR BILLIARDS

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ABSTRACT. Let γ be an orbit of the billiard flow on a convex planar billiard table; then the perpendicular part of the derivative of the billiard flow along γ is a symplectic linear map DP . This paper contains a proof of the following Franks' lemma for a residual set of convex planar billiard tables: for any closed orbit, the map DP can be perturbed freely within a neighborhood in $Sp(1)$ by a C^2 -small perturbation in the space of convex planar billiard tables.

1. INTRODUCTION

The long-term behavior of orbits of diffeomorphisms or flows is of central importance in dynamical systems. Given an orbit (or a “typical” orbit) γ , one would like to know not only where γ travels in the ambient space, but also how close-by orbits behave. The derivative of the diffeomorphism or flow along an orbit is an important object for studying the behavior of close-by orbits. For an arbitrary orbit γ , Lyapunov exponents (when they exist) describe the long-term linear behavior along γ ; for a periodic orbit, these are simply the (logarithms) of the eigenvalues of the derivative.

Given a closed orbit of a dynamical system, a natural question is how the derivative depends on the system. For example, can one prescribe the derivative along a given orbit? A Franks' lemma is a tool that allows one to freely perturb the derivative in a neighborhood (in the appropriate linear space) by perturbing the dynamical system (in the appropriate space of dynamics). The name comes from a lemma of John Franks for diffeomorphisms in [11]. This utility of this kind of result is that it relates perturbations in a complicated space (such as diffeomorphisms or flows) with perturbations in a linear space. Franks' lemmas have been proven in many different contexts: for instance, [4] for conservative diffeomorphisms, [2], [12] and [1] for symplectomorphisms, [13] and [5] for flows, [3] for conservative flows, [17] for Hamiltonians, and [9], [8], and [16] for geodesic flows. Note that more restrictive settings often need to utilize tools specific to their setting—a geodesic flow can be perturbed as a Hamiltonian flow, but the result may no longer be a geodesic flow.

This paper contains a Franks' lemma for convex planar billiards, using methods that the author developed for geodesic flows in [16]. The general idea is that for geodesic flows, the derivative of the flow along an orbit can be written in terms of Jacobi fields. Since the billiard flow is induced by the geodesic flow on the plane, Jacobi fields take a particular form for convex planar billiards, and they can be perturbed by small perturbations to the curvature of the billiard table at the points where the orbit hits the boundary. In order to maintain control of the orbit while making perturbations to its derivative, we regard only closed orbits that do not hit a point on the boundary multiple times during one period. We show that there is a residual set of convex planar billiard tables that have this property as well as a condition that can be described geometrically as “no focusing along length-3 pieces of periodic orbits.” Given a table and a periodic orbit γ

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of the billiard flow, we call the *period* of γ the number of times γ hits the boundary of the table before returning to its original vector. Then the following theorem states that the derivative of the billiard flow along a periodic orbit of period at least 3 can be freely perturbed in a small ball in the space of symplectic linear maps by making a small perturbation to the convex planar billiard table in the C^2 topology.

Theorem 1. *There exists a residual set \mathcal{R} of C^2 convex planar billiards such that for any $\alpha \in \mathcal{R}$, for any closed orbit γ of period at least 3, and for any C^2 neighborhood \mathcal{U} of α , there exists an open ball $B \subset Sp(1)$ around $DP(\gamma, \alpha)$ such that any element of B is realizable as $DP(\gamma, \tilde{\alpha})$ for some $\tilde{\alpha} \in \mathcal{U}$. Moreover, the perturbation can be supported in an arbitrarily small neighborhood of three sequential points of γ on the boundary of the table.*

2. PRELIMINARIES

A C^r billiard table is defined by a C^r embedding $\alpha : \mathbb{S}^1 \rightarrow \mathbb{R}^2$. Sometimes it is useful to consider arc-length parametrization, which gives a map $\alpha : \mathbb{R}/\ell\mathbb{Z} \rightarrow \mathbb{R}^2$, where ℓ is the length of the image curve. By the Jordan Curve Theorem, the image of α separates the plane into two regions, one of which is bounded—the closure of this region is defined to be the billiard table \mathcal{D} . A C^2 billiard table \mathcal{D} is convex if $\frac{d^2}{dt^2}\alpha \neq 0$ and points into the interior of \mathcal{D} everywhere; we follow the convention that for such tables the curvature is positive. Let $S_{\mathcal{D}}\mathbb{R}^2$ denote the unit tangent bundle on \mathbb{R}^2 restricted to $\mathcal{D} \subset \mathbb{R}^2$, and let $S\mathcal{D} = S_{\mathcal{D}}\mathbb{R}^2 / \sim$, where two points (x, v) and (x, v') are equivalent if $x \in \partial\mathcal{D}$ and v' is the reflection of v across the line tangent to α at x . Then the billiard flow $\varphi_t : S\mathcal{D} \times \mathbb{R} \rightarrow S\mathcal{D}$ is induced by the geodesic flow on the plane. Since the dynamics of the billiard flow is not dependent on where or how the table is positioned in the plane, we will consider the set of curves modulo rigid motions of the plane and reparametrizations of the defining curve. Let \mathcal{B} denote the set of all such C^2 convex billiard tables, modulo rigid motions and reparametrizations, so that elements of \mathcal{B} are equivalence classes $[\alpha]$ (see [6]).

A billiard table α of class C^2 has a normal bundle $(\alpha(t), \mathbf{n}(t))$, where $\mathbf{n}(t)$ is the unit normal vector to $\alpha'(t)$ pointing into \mathcal{D} , of class C^1 . For $\varepsilon > 0$, consider the following tubular neighborhood of the image of α :

$$N_\varepsilon(\alpha) = \{\alpha(t) + \lambda\mathbf{n}(t) \mid -\varepsilon < \lambda < \varepsilon\} \subset \mathbb{R}^2.$$

Definition 2. *Two equivalence classes of convex billiard tables $[\alpha], [\beta] \in \mathcal{B}$ are (C^0) ε -close if there exist representatives $\alpha \in [\alpha]$ and $\beta \in [\beta]$ such that the image of β is contained in $N_\varepsilon(\alpha)$ and the canonical projection $\beta(t) \mapsto \alpha(t)$ is a diffeomorphism.*

For two such representatives we can write $\beta(t) = \alpha(t) + \lambda(t)\mathbf{n}(t)$, with λ a C^2 periodic function.

Definition 3. *$[\alpha]$ and $[\beta]$ are ε C^2 -close if there are representatives α and β such that $\beta(t) = \alpha(t) + \lambda(t)\mathbf{n}(t)$ with $\|\lambda\|_{C^2} < \varepsilon$.*

The behavior of a flow along an orbit can be studied by picking hypersurfaces transverse to the flow and considering the maps that take one section to the next via the flow. In the context of a billiard flow, there are two natural choices of transversals—one that regards the billiard flow as induced by the geodesic flow on the plane, and a second that uses the property that there is a global section of the billiard flow provided by the boundary of the table.

Following the first method, given an orbit γ of the billiard flow, pick surfaces Σ_t along γ in $S\mathcal{D}$ that are perpendicular to the flow direction. For any two of these (say, at $t = 0$ and $t = 1$), one gets a Poincaré map $P : \Sigma_0 \rightarrow \Sigma_1$ via the billiard flow, defined in a neighborhood of the origin (the point where γ intersects the surface). We are interested in the derivative of this map at the origin, $DP : T_0\Sigma_0 \rightarrow T_0\Sigma_1$. The map DP can be written nicely in Jacobi coordinates (U, V) on $T_0\Sigma_t$, where U is the horizontal component of $T\mathcal{S}\mathcal{D}$ restricted to $T_0\Sigma_t$ and V is the

vertical component of TSD restricted to $T_0\Sigma_t$. In these coordinates, the Poincaré map between two interior sections is given by

$$T := DP = \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix}$$

where τ is the distance between the two sections. At a point of reflection (on the boundary) the derivative is given by

$$R := DP = \begin{bmatrix} -1 & 0 \\ \frac{2k_p}{\cos \theta_p} & -1 \end{bmatrix}$$

where k_p is the curvature at the point p of reflection (positive, since \mathcal{D} is convex), and $\theta_p \in (-\pi/2, \pi/2)$ is the angle $\dot{\gamma}$ makes with the unit normal vector \mathbf{n} at p . (See, for instance, [7].¹) Since any linear Poincaré map in these coordinates is a product of matrices of the above types, both of which have determinant 1, all such maps are elements of $SL(2) = Sp(1)$.

A billiard flow also has a global section that every orbit must intersect repeatedly. Let $\mathcal{C} = \mathbb{R}/\ell\mathbb{Z} \times (-\pi/2, \pi/2)$ be the cylinder with first coordinate identified with the boundary of \mathcal{D} under α and second coordinate giving the angle of a vector with the normal vector \mathbf{n} . Then the billiard flow induces a map $T : \mathcal{C} \rightarrow \mathcal{C}$ taking a point (p, θ) to the next boundary point hit by the trajectory determined by (p, θ) , known as the billiard map. This map is more commonly studied than the Poincaré map defined above, but their derivatives contain equivalent information: at points along the boundary, DP and DT are related by the equations

$$U = \cos \theta dr \quad \text{and} \quad V = k dr + d\theta,$$

where r is the arc-length parameter and θ is the angle with the unit normal \mathbf{n} ([7]).

3. A RESIDUAL SUBSET OF CONVEX BILLIARDS

The proof of Theorem 1 requires that the initial billiard table have the property that there is no focusing along length-3 segments of periodic orbits. In this section, we define a subset of convex planar billiards that has this property, and show that it is a residual set. Recall that the *period* of a periodic orbit γ of the billiard flow refers to the number of times γ hits the boundary of the table before returning to its original point in $S\mathcal{D}$, and let T_α be the billiard map corresponding to the table defined by α .

Definition 4. *Let \mathcal{R} be the set of C^2 convex billiards with the following properties:*

- (1) *for each $N > 0$, there are a finite number of orbits of period N and they are all nondegenerate;*
- (2) *along each periodic orbit, each point of the boundary is hit at most once;*
- (3) *at each point p along a periodic orbit*

$$\frac{2k_p}{\cos \theta_p} \neq \frac{1}{\tau_-} + \frac{1}{\tau_+},$$

where τ_- is the length of the incoming piece of orbit and τ_+ is the length of the outgoing piece of orbit.

Lemma 5. *\mathcal{R} contains a C^2 -residual subset of \mathcal{B} .*

Proof. The set $\mathcal{U}_N = \{\alpha \in \mathcal{B} : \text{for all } n|N, \text{ every period-}n \text{ orbit of } T_\alpha \text{ is non-degenerate}\}$ is C^2 open and C^2 dense in \mathcal{B} by Theorem 1 in [6].

¹[18] and [10] are also good references, but they define θ as the angle with the tangent line to the table rather than the inward normal. The resulting statements require a small translation to match the ones given here.

Let $\mathcal{U}'_N = \{\alpha \in \mathcal{U}_N : \text{for all } n|N, \text{ period-}n \text{ orbits have no multiple hit points}\}$. Theorem 1 in [14] shows that condition 2 is C^2 dense, since it is C^∞ dense. To show that \mathcal{U}'_N is C^2 open, recall that a period- n orbit corresponds to a critical point of the function

$$(\mathbb{S}^1)^n \xrightarrow{\alpha^n} \mathbb{R}^{2n} \xrightarrow{d} \mathbb{R},$$

where d assigns to each n -tuple of points on the boundary of the table α the distance traveled by the straight-line path connecting each point in order ([15]). Since $D_p(d \circ \alpha^n) = D_{\alpha^n(p)}d \circ D_p\alpha^n$, the critical points of this map move continuously in $(\mathbb{S}^1)^n$ with α in the C^1 topology. As the set $\{(x_1, \dots, x_n) : x_i \neq x_j \text{ for all } i, j\}$ is open in $(\mathbb{S}^1)^n$, the set of billiard tables in \mathcal{B} without multiple hit points along a finite number of closed orbits is open in the C^1 (and therefore also C^2) topology.

Let $\mathcal{U}''_N = \{\alpha \in \mathcal{U}'_N : \text{for all } n|N, \frac{2k_p}{\cos \theta_p} \neq \frac{1}{\tau_-} + \frac{1}{\tau_+} \text{ along per-}n \text{ orbits}\}$. Because $\alpha \in \mathcal{U}'_N$, there are a finite number of period- n orbits. Thus, there are a finite number of points on $\alpha(\mathbb{S}^1)$ to consider. Since we want to avoid a particular value of the curvature at each of these points, \mathcal{U}''_N is also C^2 -open and C^2 -dense. Then taking the intersection $\bigcap_{N=1}^\infty \mathcal{U}''_N$ gives a C^2 -residual subset with the desired properties. \square

The statement $d_p = \frac{2k_p}{\cos \theta_p} \neq \frac{1}{\tau_-} + \frac{1}{\tau_+}$ in Definition 4 can be interpreted geometrically as there being no focusing along this length-3 piece of orbit. To see this, observe that

$$T_- R_p T_+ = \begin{bmatrix} -1 + \tau_- d_p & -\tau_+ - \tau_- + \tau_+ \tau_- d_p \\ d_p & -1 + \tau_+ d_p \end{bmatrix},$$

so that the Jacobi field defined by $J(0) = 0$, $J'(0) = 1$ becomes

$$\begin{bmatrix} J(\tau_- + \tau_+) \\ J'(\tau_- + \tau_+) \end{bmatrix} = T_- R_p T_+ \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\tau_+ - \tau_- + \tau_+ \tau_- d_p \\ -1 + \tau_+ d_p \end{bmatrix}.$$

The Jacobi field is focused if $J(\tau_- + \tau_+) = 0$, or $d_p = \frac{1}{\tau_-} + \frac{1}{\tau_+}$.

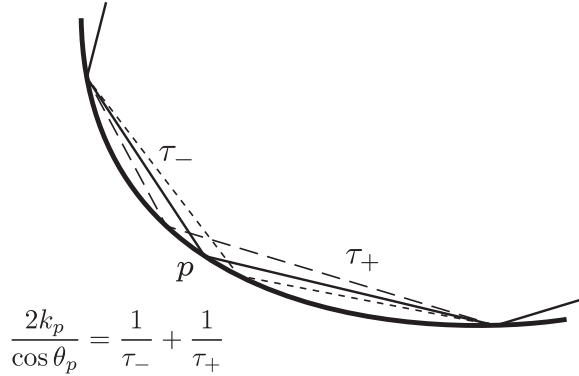


FIGURE 1. Focusing along a billiard trajectory.

We require a billiard table to be in the set \mathcal{R} in order to apply the proof techniques for Theorem 1 along a periodic orbit of period at least 3. This may be in part due to the techniques used, but the following example shows that it is not always possible to freely perturb the derivative of the billiard flow along *any* closed orbit on *any* table. Consider a period-2 orbit on a table where the curvature at each point of the boundary that the orbit hits is equal to $1/\tau$, where τ is the distance the orbit travels from one point on the boundary to the next. Making a perturbation of the billiard table such that γ remains a periodic orbit can only change three quantities: the

curvatures k_1 and k_2 at p_1 and p_2 , and the distance τ . By symmetry, however, perturbing the curvature at p_1 has the same effect to DP as perturbing the curvature at p_2 , so that there are at most two directions in $Sp(1)$ that one can move DP by perturbations of the billiard table (a second direction would come from perturbing τ). Since $\dim Sp(1) = 3$, however, this is a non-trivial restriction on how DP can be perturbed in $Sp(1)$.

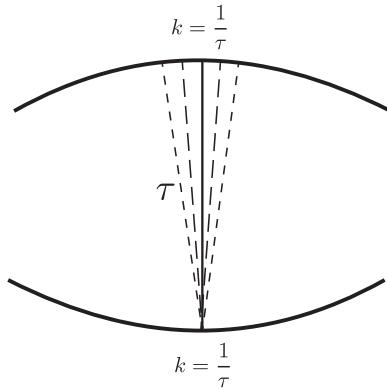


FIGURE 2. An example in which, even allowing for perturbations to τ , one cannot freely perturb DP in $Sp(1)$.

4. PROOF OF THEOREM 1

The following arguments are based on methods developed in [16] for geodesic flows. In that context, the arguments produce a perturbation size that was uniform over all length-1 geodesic on the manifold. In this context, due to the fact that length refers to the number of times an orbit segment hits the boundary of the billiard table (rather than the length of the segment in the billiard flow), there is no such uniformity. In particular, as the τ_i get smaller (e.g., take a sequence of orbits that accumulate on the boundary of the table), the effects of the curvature perturbations get smaller.

Fix α , \mathcal{U} , and γ . In order to prove Theorem 1, it suffices to make perturbations to just one length-3 piece of the periodic orbit. Then the theorem is proved by using following lemma to perturb the linear Poincaré map along a length-3 piece of orbit:

Lemma 6. *Let $\alpha \in \mathcal{B}$, and γ be an orbit segment of length 3 with $p_1 \neq p_3$ and $d_2 \neq \frac{1}{\tau_2} + \frac{1}{\tau_3}$. Then for any C^2 neighborhood \mathcal{U} of α there exists an open ball $B \subset Sp(1)$ about $DP(\gamma, \alpha)$ such that any element of B is realizable as $DP(\gamma, \tilde{\alpha})$ for some $\tilde{\alpha} \in \mathcal{U}$.*

Proof. For a fixed neighborhood \mathcal{U} of α , there is a ball of some radius ε , $\inf_{q \in \alpha} \{\frac{1}{2}k_\alpha(q)\} > \varepsilon > 0$, contained in \mathcal{U} . By Lemma 7 below, this means that we can perturb the curvature at any point by size up to $\varepsilon > 0$ while preserving γ as a piece of orbit.

Now we wish to show that any element of $B_\delta(DP(\gamma, \alpha))$ (for some δ) can be realized as the linear Poincaré map along γ by making size ε or smaller perturbations of the curvature of the table at the three points γ hits. The map $A = DP(\gamma, \alpha)$ can be written as

$$A = \begin{bmatrix} -1 & 0 \\ d_3 & -1 \end{bmatrix} \begin{bmatrix} 1 & \tau_3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ d_2 & -1 \end{bmatrix} \begin{bmatrix} 1 & \tau_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ d_1 & -1 \end{bmatrix} \begin{bmatrix} 1 & \tau_1 \\ 0 & 1 \end{bmatrix}$$

where $d_i = \frac{2k_i}{\cos \phi_i}$. Perturbing the curvatures k_1 , k_2 , and k_3 is, in effect, perturbing d_1 , d_2 , and d_3 since the tangent lines and therefore the angles ϕ_i are preserved by the perturbations of

Lemma 7. We want to make sure that these three perturbations move the map DP in distinct directions in $Sp(1)$. By Lemma 8 below, the matrices $\frac{\partial}{\partial d_1}DP$, $\frac{\partial}{\partial d_2}DP$, and $\frac{\partial}{\partial d_3}DP$ are linearly dependent only if $d_2 = \frac{1}{\tau_2} + \frac{1}{\tau_3}$; but $d_2 \neq \frac{1}{\tau_2} + \frac{1}{\tau_3}$ by hypothesis.

Let $\mathcal{K} = \{(k_{p_1}, k_{p_2}, k_{p_3})\} \cong \mathbb{R}^3$ be the space of curvatures at the points p_1, p_2 , and p_3 respectively, and let $\Phi : \mathcal{K} \rightarrow Sp(1)$ be the map that assigns to each $(k_{p_1}, k_{p_2}, k_{p_3})$ the product DP . By the arguments above, this map has full rank at $k = (k_1, k_2, k_3)$, so by the Inverse Function Theorem Φ is a local diffeomorphism and the image of a neighborhood of k under Φ contains an open ball about $DP(\gamma, \alpha)$. \square

Lemma 7. *Let p be a point on the billiard table α with curvature $k_\alpha(p)$. Then there exists $\varepsilon_\alpha > 0$ such that for any $\varepsilon_\alpha > \varepsilon > 0$, for any $k \in \mathbb{R}$ with $|k - k_\alpha(p)| < \varepsilon$, and for any neighborhood $V \subset \alpha(\mathbb{S}^1)$ of p , there is a perturbation $\tilde{\alpha}$ with the following properties:*

- (1) $k_{\tilde{\alpha}}(p) = k$,
- (2) $\alpha = \tilde{\alpha}$ outside of V ,
- (3) $\tilde{\alpha} \in \mathcal{B}$ (i.e. is a C^2 convex billiard table), and
- (4) $\|\alpha - \tilde{\alpha}\|_{C^2} < \varepsilon$.

Proof. Let $\varepsilon_\alpha = \inf_{q \in \alpha} \{\frac{1}{2}k_\alpha(q)\}$. Fix a parametrization of α with $\alpha(0) = p$, and let $\delta > 0$ such that $\alpha(-\delta, \delta) \subset V$. Let $\psi : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function with the following properties:

- (1) $\text{supp}(\psi) = (-\delta, \delta)$
- (2) $\|\psi\|_{C^2} \leq \varepsilon$
- (3) $\psi(0) = \psi'(0) = 0$
- (4) $\psi''(0) = k - k_\alpha(p)$

Let $\tilde{\alpha}(t) = \alpha(t) + \psi(t)\mathbf{n}(t)$. Then

$$k_{\tilde{\alpha}}(p) = k_\alpha(p) + \psi''(0) = k,$$

and $\|\alpha - \tilde{\alpha}\|_{C^2} = \|\psi\|_{C^2} \leq \varepsilon$. Since $\varepsilon < \varepsilon_\alpha$, we have $\tilde{\alpha} \in \mathcal{B}$. Moreover, since $\psi(0) = \psi'(0) = 0$, the table $\tilde{\alpha}$ still goes through the point p and has the same tangent line as α at p . \square

Lemma 8. *The matrices $A = \frac{\partial}{\partial d_1}DP$, $B = \frac{\partial}{\partial d_2}DP$, and $C = \frac{\partial}{\partial d_3}DP$ are linearly dependent if and only if $d_2 = \frac{1}{\tau_2} + \frac{1}{\tau_3}$.*

Proof. Multiplying out DP and taking derivatives yields

$$A = \begin{bmatrix} \tau_2 + \tau_3 - \tau_2\tau_3d_2 & (\tau_2 + \tau_3 - \tau_2\tau_3d_2)\tau_1 \\ 1 - (d_2 + d_3)\tau_2 - \tau_3d_3 + \tau_2\tau_3d_2d_3 & (1 - (d_2 + d_3)\tau_2 - \tau_3d_3 + \tau_2\tau_3d_2d_3)\tau_1 \end{bmatrix}$$

$$B = \begin{bmatrix} (1 - \tau_2d_1)\tau_3 & (\tau_1 + \tau_2 - \tau_1\tau_2d_1)\tau_3 \\ 1 - \tau_2d_1 - \tau_3d_3 + \tau_2\tau_3d_1d_3 & (1 - \tau_2d_1 - \tau_3d_3 + \tau_2\tau_3d_1d_3)\tau_1 + \tau_2 - \tau_2\tau_3d_3 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 \\ c & c\tau_1 + \tau_2 + \tau + 3 - \tau_2\tau_3d_2 \end{bmatrix},$$

where $c = 1 - (d_1 + d_2)\tau_3 - \tau_2d_1 + \tau_2\tau_3d_1d_2$. Linear dependency requires $\alpha A + \beta B + \gamma C = 0$, with α, β, γ not all 0. This yields four equations, which are satisfied simultaneously if and only if $d_2 = \frac{1}{\tau_2} + \frac{1}{\tau_3}$, $\alpha = \frac{\gamma\tau_3^2}{\tau_2^2}$, and $\beta = 0$ (also if any $\tau_i = 0$, but these cases describe degenerate orbits and thus are not relevant). \square

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