MINIMAL FREE RESOLUTIONS OF COMPLETE BIPARTITE GRAPH IDEALS

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Abstract. This paper gives an explicit construction for the minimal free resolution of a complete bipartite graph ideal. This yields a combinatorial formula for the Betti numbers and projective dimension of complete bipartite graph ideals.

The question of how to compute the minimal free resolution of an ideal in a polynomial ring $R = k[x_1, ..., x_n]$ and the information it provides are very important in commutative algebra. It is theoretically possible, via algorithmic methods, to compute any free resolution with the computer program Macaulay2. We are interested, however, in finding an explicit construction of the minimal free resolution for ideals that have a combinatorial nature. Using results from [BS], this paper presents such a construction for complete bipartite graph ideals.

This paper gives an explicit description of the minimal free resolution of complete bipartite graph ideals by providing a topological cell complex whose corresponding cellular complex is minimal (this topological cell complex is not simplicial and so is not the usual cell complex associated to monomial ideals). This approach not only gives a simple explicit description of the minimal free resolution but provides, as a corollary, a combinatorial description of the betti numbers for these ideals. In independent work, Sean Jacques [SJ, Theorem 5.2.4] gives the same combinatorial description of the betti numbers for complete bipartite ideals. Recently, Ha and Van Tuyl [HV] also have given combinatorial descriptions of the betti numbers for certain types of edge ideals, including complete bipartite ideals as these are iteratively splittable. These works, however, do not provide an explicit description of the full minimal free resolution, and thus the present paper is of independent interest due to the topological cell complex construction whose ideas may prove useful in other contexts.

Graph ideals are monomial, square-free ideals generated by quadrics that have a natural correspondence to combinatorial graphs. Let G be a graph with vertex set $V = \{x_1, x_2, .., x_n\}$ and edge set E. Then in the polynomial ring $R = k[x_1, x_2, .., x_n]$ over a field k, we define the graph ideal $I(G)$ as having generators $x_i x_j$ where $(x_i, x_j) \in E$. In this paper we focus on the subset of graph ideals called complete bipartite graph ideals. A *bipartite graph* is a graph G with vertex set V that can be partitioned into two subsets A and B such that every edge has one vertex in A and the other in B, and is called *complete* if given any vertex $a \in A$, $(a, b) \in E$ for every $b \in B$. Thus a general complete bipartite graph ideal is of the form $I(G) = \langle a_1, .., a_n\rangle \cap \langle b_1, .., b_m\rangle$, where $\{a_1, ..., a_n, b_1, ..., b_m\} = \{x_1, ..., x_{n+m}\}$. This corresponds to the graph G with n vertices in $A \subset G$ and m vertices in $B \subset G$.

In the rest of this paper we will label variables as $a_1, ..., a_n, b_1, ..., b_m$ instead of $x_1, ..., x_{n+m}$ for convenience.

Square-free monomial ideals, and thus graph ideals, have the additional property that they can be decomposed into an intersection of prime ideals that are generated by sets of variables. We refer to these minimal associated prime ideals as *irreducible primes*. They play an essential role in the structure of graph ideals, and can be computed graph theoretically via the minimal vertex covers of the supporting graph G.

The codimension of a graph ideal I, written $\text{codim}(I)$, is the smallest codimension of its irreducible primes. Since the irreducible primes of a graph ideal are generated by sets of variables, the codimension of these is simply the number of generators. This value is used in an important classification: if an ideal I in a polynomial ring R satisfies $\text{pd}(R/I) = \text{codim}(I)$, where $\text{pd}(R/I)$ is the projective dimension of R/I, it is said to be Cohen-Macaulay. For further classification of graph ideals that are Cohen-Macaulay, the reader may refer to [F] and [V]. In this paper we are interested not only in the smallest codimension of the irreducible primes but also in the largest.

Definition 1. *Let* $\text{mcip}(I)$ *be the maximal codimension of the irreducible primes of the ideal* I.

Since there are only two irreducible primes for any complete bipartite graph ideal, we consider *all* of the irreducible primes of I when we consider both its codimension and maximal codimension. Also, if an ideal of this form is Cohen-Macaulay, then $\text{mcip}(I) = \text{codim}(I)$, and a result later in this paper shows that $\text{codim}(I) = 1$.

Our construction of the minimal free resolution rests on a labelled topological cell complex $X(I)$. We build this cell complex in two steps:

(1) Let $A = \{a_1, ..., a_n\}$ and $B = \{b_1, ..., b_m\}$. Construct l-cells with labels $a_i b_j \prod_{k=1}^{l} u_k$, where $a_i \in A$, $b_j \in B$, $u_k \in A \cup B$, and every variable in the label is distinct.

(2) Attach the l-cell labelled α to every $(l-1)$-cell labelled β that satisfies $\alpha = u_k\beta$ where $u_k \in A \cup B$, so that $\partial(\alpha) = \{\beta \mid \alpha = u_k\beta\}$.

An example is given below.

Example 2. *The cell complex* $X(I)$ *corresponding to the ideal* $I = \langle a_1, a_2\rangle \cap \langle b_1, b_2, b_3, b_4\rangle$*, as given by the construction above, is shown in Figure 1. This complex has eight* 0*-cells, labelled with the generators of* I*, sixteen* 1*-cells, fourteen* 2*-cells, six* 3*-cells, and one* 4*-cell with label* $a_1a_2b_1b_2b_3b_4$*.*

There is a standard chain complex associated with any topological cell complex, given by

$$0 \longrightarrow C_p(X;k) \xrightarrow{\partial_p} C_{p-1}(X;k) \xrightarrow{\partial_{p-1}} \cdots \xrightarrow{\partial_1} C_0(X;k) \xrightarrow{\partial_0} 0,$$

where $C_i(X;k)$ is the k-module generated by the i-faces of X. In the example above, $C_3(X;k) \cong k^6$ since there are six 3-cells.

The cell labels given above can easily be incorporated into the chain complex on $X(I)$. For each generating cell in $C_i(X;k)$, use its label as a generator of a free module in $R = k[a_1, ..., a_n, b_1, ..., b_m]$. Then the boundary map ∂ is given in step (2) of the construction above.

For notational convenience, however, it will be advantageous to use an alternate representation for the labels of the cell complex. We can represent each label

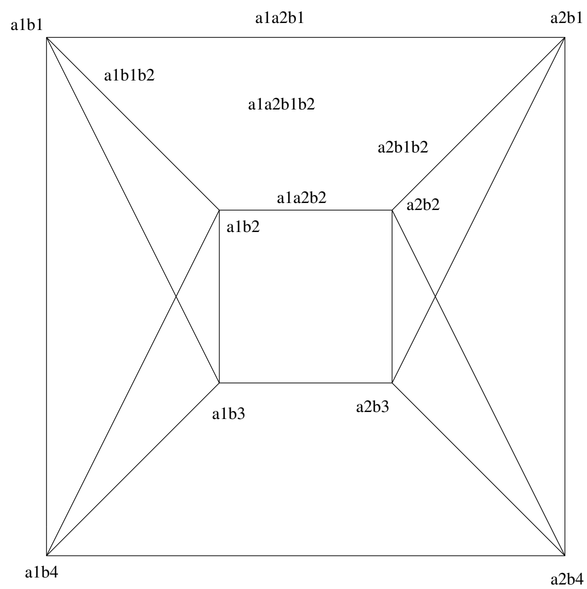

FIGURE 1. $X(I)$ for $I = \langle a_1, a_2\rangle \cap \langle b_1, b_2, b_3, b_4\rangle$, with selected labels

equivalently as a $(0,1)$-vector with $(n+m)$ entries, where a 1 in the i^{th} position signifies that the variable x_i is found in the label, as notated in [BS]. Then every cell, also called a face F of $X(I)$, has an associated $(0,1)$-vector called the degree $\mathbf{a}_F$ of F. Thus in the example above the 1-cell with label $a_1a_2b_2$ has degree $(1,1,0,1,0,0)$. Incorporating these labels into the chain complex yields the cellular complex $\mathbf{F}_X$ described in [BS], as this puts a $\mathbb{Z}^n$-grading on the free R-modules labelled by the face degrees. For each degree $\mathbf{b} \in \mathbb{Z}_2^n$, let $X_{\preceq \mathbf{b}}$ be the subcomplex of X on the vertices of degree $\preceq \mathbf{b}$.

The following results from [BS] establish when the cellular complex $\mathbf{F}_X$ is a free resolution and when it is minimal.

Proposition 3. [BS, Proposition 1.2] *The complex $\mathbf{F}_X$ is a free resolution of I if and only if $X_{\preceq \mathbf{b}}$ is acyclic over k for all degrees $\mathbf{b}$.*

Remark 4. [BS, Remark 1.4] *The cellular complex $\mathbf{F}_X$ is a minimal resolution if and only if any two comparable faces $F' \subset F$ of the complex X have distinct degrees $\mathbf{a}_F \neq \mathbf{a}_{F'}$.*

The main theorem, below, establishes that the cellular complex $\mathbf{F}_X$ given above is equivalent to the minimal free resolution of I. Note that, as the results in Bayer and Sturmfels [BS] are independent of the field k, so are these results.

Theorem 5. *Let $I = \langle a_1, .., a_n\rangle \cap \langle b_1, .., b_m\rangle$ be a complete bipartite graph ideal. Then the cellular complex $\mathbf{F}_X$ constructed above is equivalent to the minimal free resolution of I.*

Proof. First, by part (1) in the construction above, any degree $\mathbf{b} \in \mathbb{Z}$ corresponds to a unique cell in the complex $X(I)$, and $X_{\preceq \mathbf{b}}$ is the full subcomplex of this cell.

Thus $X_{\preceq \mathbf{b}}$ is homeomorphic to the p-dimensional ball, where p is two less than the number of ones in the degree $\mathbf{b}$, and therefore acyclic. Then, by Proposition 1.2 in [BS], $\mathbf{F}_X$ is a free resolution of I. Since all cells in X have distinct labels, the free resolution $\mathbf{F}_X$ is also minimal by Remark 1.4 in [BS]. Since the minimal free resolution of a homogeneous ideal is unique, $\mathbf{F}_X$ is the minimal free resolution of I. □

Corollary 6. *The $(k+1)^{th}$ Betti number of the minimal free resolution of R/I is equivalent to the number of k-cells in the cell complex constructed above, and is given by*

$$b_{k+1} = \#(k - cells) = \sum_{j=1}^{k+1} \binom{n}{j} \binom{m}{k-j+2}.$$

Proof. That $b_{k+1} = \#(k - cells)$ follows directly from the fact that $\mathbf{F}_X$ is the minimal free resolution of I. Now we count up the number of k-cells in the complex $X(I)$. By construction, every k-cell will have a label with $k+2$ variables, one of which must be an a and another of which must be a b. The number of labels for k-cells in which there are x-number of a's is $\binom{n}{x}\binom{m}{k-x+2}$. Thus, if we sum this up from the minimum $x = 1$ to the maximum $x = k+1$ we get

$$\#(k - cells) = \sum_{j=1}^{k+1} \binom{n}{j} \binom{m}{k-j+2}.$$

□

From the explicit formula for the Betti numbers we also get an explicit formula for the projective dimension.

Corollary 7. *The projective dimension of R/I is given by*

$$\mathrm{pd}(R/I) = \mathrm{codim}(I) + \mathrm{mcip}(I) - 1 = m + n - 1.$$

Proof. This follows directly from Corollary 6, which states that $b_{k+1} = \#(k - cells)$, and the fact that the projective dimension is the index of the last non-zero Betti number. Since the largest cell in the complex $X(I)$ has label $a_1 ... a_n b_1 ... b_m$, and this corresponds to an $(m+n-2)$-cell, the last non-zero Betti number is $b_{m+n-1} = \#((m+n-2) - cells) = 1$. □

Example 8. *We use the ideal $I = \langle a_1, a_2 \rangle \cap \langle b_1, b_2, b_3, b_4 \rangle$ from the previous example. The third Betti number of the minimal free resolution of R/I is equivalent to the number of 2-cells in $X(I)$, which is $\sum_{x=1}^{3} \binom{2}{x}\binom{4}{4-x} = \binom{2}{1}\binom{4}{3} + \binom{2}{2}\binom{4}{2} + \binom{2}{3}\binom{4}{1} = 8 + 6 + 0 = 14$. Similar computations for the other numbers of k-cells show that, by the theorem above, the minimal free resolution of R/I is*

$$0 \to R^1 \xrightarrow{\phi_4} R^6 \xrightarrow{\phi_3} R^{14} \xrightarrow{\phi_2} R^{16} \xrightarrow{\phi_1} R^8 \xrightarrow{\phi_0} R^1 \to 0.$$

The projective dimension of R/I, by Corollary 7, is $\mathrm{pd}(R/I) = m + n - 1 = 2 + 4 - 1 = 5$. Also, the map ϕ_4 can be found using the boundary map ∂_4 of the *chain complex on $X(I)$, and looks like:*

$$\begin{array}{c} \\ a_1a_2b_1b_2b_3 \\ a_1a_2b_1b_2b_4 \\ a_1a_2b_1b_3b_4 \\ a_1a_2b_2b_3b_4 \\ a_1b_1b_2b_3b_4 \\ a_1b_1b_2b_3b_4 \end{array} \begin{array}{c} a_1a_2b_1b_2b_3b_4 \\ \left[\begin{array}{r} b_4 \\ -b_3 \\ b_2 \\ -b_1 \\ -a_2 \\ a_1 \end{array}\right] \end{array}$$

where the signs come from the orientation assigned to $X(I)$.

An ideal is called *linear* if all of its syzygies are linear. In particular, I is linear if the matrices in the minimal free resolution have only linear entries. The linearity of the ideal I follows from our construction of the minimal free resolution.

Corollary 9. *I is linear and the only non-zero i^{th} graded betti number occurs in degree $i+1$ and is given by the combinatorial formula for the i^{th} total betti number.*

Proof. The matrix entries of the minimal free resolution come from the boundary maps of the labelled cell complex $X(I)$ by Theorem 5. Since every boundary is linear, all of the matrix entries are linear and thus I is linear. Further, since all generators of I are degree two and the matrices in the minimal free resolution are all linear, the graded betti numbers are zero, except in degree $i + 1$ and in this degree are equal to the full betti numbers. □

That these ideals are linear is a strong result, as many properties of linear ideals are known. One reference for recent results on linear ideals is [EGHP].

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